Quantitative Finance Qualifying Exam
2018 Summer

INSTRUCTIONS

You have 4 hours to do this exam.

Reminder: This exam is closed notes and closed books. No electronic devices are permitted. Phones must be turned completely off for the duration of the exam.

PART 1: Do 2 out of problems 1, 2, 3.
PART 2: Do 2 out of problems 4, 5, 6.
PART 3: Do 2 out of problems 7, 8, 9.
PART 4: Do 2 out of problems 10, 11, 12.

All problems are weighted equally.
On this cover page write which eight problems you want graded.

Problems to be graded:

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty.
Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (PRINT CLEARLY), ID number:

Signature

Stony Brook University
Applied Mathematics and Statistics

1. Spot Rate Computation and Applications

You are given the following information. Assume annual compounding throughout.

- A 1-year zero-coupon bond with a face value of $10,000 sells at a discount of $9,900.
- A 2-year bond with a face value of $10,000 and an annual coupon of $300 sells at premium of $10,100.
- A 3-year bond with a face value of $100,000 and an annual coupon of $3,100 sells at a discount of $99,156.

Solve for the following:

a) Spot Curve: Using the market data, above bootstrap the 3-year spot curve.

b) Bond Valuation: Using the spot rates computed above compute the price of a 3-year bond with a face value of $1,000,000 and annual coupon of $4,000.

c) Forward Rate: Compute the forward rate $f_{2,3}$.

2. Market Portfolio

Consider the following simple quadratic program representing the portfolios on the Capital Market Line (CML) and its solution to proportionality. The parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, are the returns' mean vector and covariance matrix, respectively, and r_f is the risk-free rate. The value of $\lambda \geq 0$ parameterizes the CML:

$$\min\left\{ \tfrac{1}{2}\mathbf{x}^T \boldsymbol{\Sigma}\, \mathbf{x} - \lambda(\boldsymbol{\mu} - r_f)^T \mathbf{x} \right\}$$

Assume the Capital Asset Pricing Model (CAPM), *i.e.*, at time t for an asset i with return $r_i(t)$, market M with return $r_M(t)$ and risk-free rate r_f, and mean-zero, uncorrelated error terms $\varepsilon_i(t)$, the following expression holds

$$r_i(t) - r_f = b_i(r_M(t) - r_f) + e_i(t)$$

Let $\mu_M = E[r_M]$ and $\sigma_M^2 = \mathrm{Var}[r_M]$.

a) Express the mean vector $\boldsymbol{\mu}$ in terms of the CAPM parameters.

b) Express the covariance matrix $\boldsymbol{\Sigma}$ in terms of the CAPM parameters.

c) Show that the covariance matrix $\boldsymbol{\Sigma}$ is positive definite.

d) Show that the value of x_i, the allocation of asset i in the market portfolio, is proportional to

$$x_i \; \mu \; \frac{b_i}{\mathrm{Var}[e_i]}$$

3. Options

Solve the following stock options problems in terms of vanilla European puts and calls and, if required, a cash position.

a) For a non-dividend-paying stock, derive the expression for put-call parity.

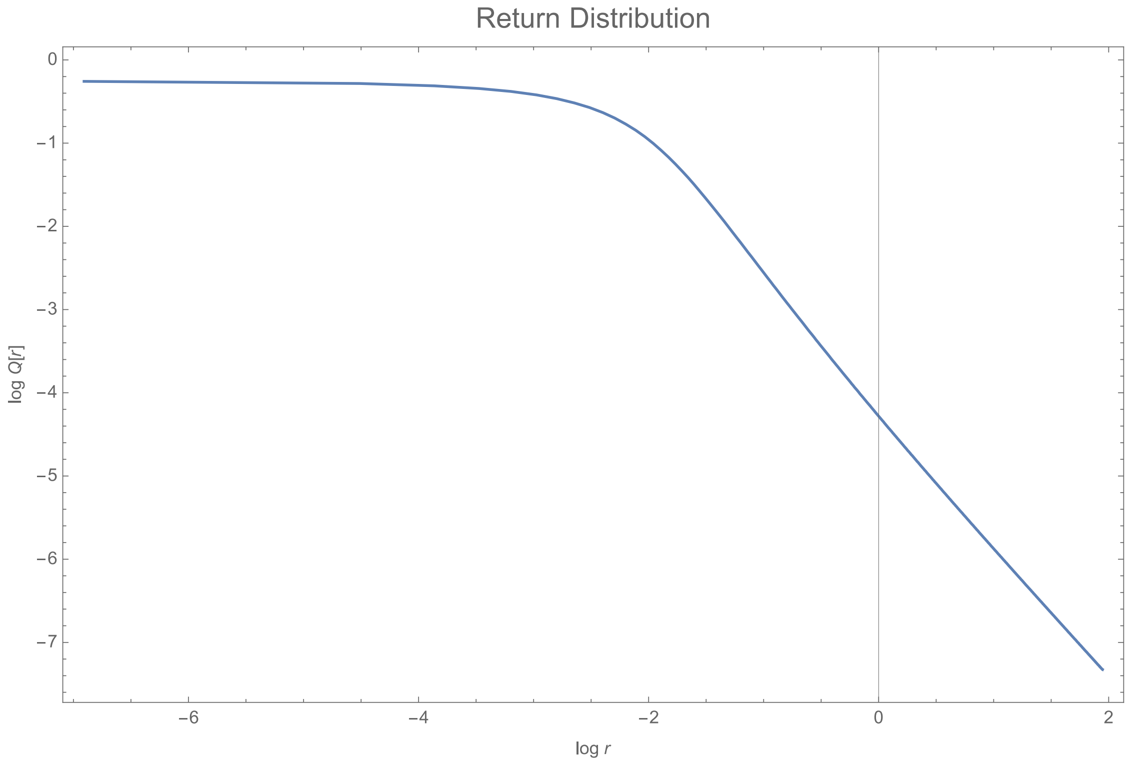
An investor owns a stock with current price S(t). The company in question is facing a lawsuit where if it wins, the price will increase dramatically and if it loses the price will decline dramatically. The judge deciding the suit has indicated that she will issue her decision on or before time T. The outcome is uncertain and the investor believes the company is as likely to win as lose the suit.

b) Construct an options portfolio that will have the potential for the investor to profit regardless of the judge's decision.

c) Under what conditions will this portfolio fail to realize a profit for the investor?

4. Power Law Model

We wish to investigate the lower tail of a return distribution. Let $Q(r) = \text{Prob}[R \geq r]$ denote the *survival function* of the return r. A plot of the log survival function for log r for $r \geq 0$ is shown below.



a) Does the distribution of r display at any point evidence that the tail of the distribution follows a power law? Explain what you looked for to determine this.

b) If so, at what point does that behavior emerge? Explain your answer.

c) If there is evidence of a power law in the tail, estimate its exponent. Employ a simple visual approximation but explain how you accomplished it. If not, hypothesize a reasonable return distribution.

d) Based on your work above, define to proportionality the PDF and CDF of the upper tail in the power law region.

e) What can you say about the existence of the moments of the distribution based on the work above? Explain you answer.

5. Markowitz Optimization

Assume that returns follow a multivariate Normal distribution with mean vector $\boldsymbol{\mu}$, positive-definite covariance matrix $\boldsymbol{\Sigma}$, and risk-free rate r_f. The mean-variance portfolio optimization with unit capital is the quadratic program below. Note that both long and short positions are allowed in this instance.

$$\mathcal{M} = \min_{\mathbf{x}} \left\{ \frac{1}{2}\mathbf{x}^T \boldsymbol{\Sigma}\, \mathbf{x} - \lambda\, (\boldsymbol{\mu}^T - r_f)^{\boldsymbol{T}} \mathbf{x} \mid \mathbf{1}^T \mathbf{x} = 1 \right\}$$

The risk-reward trade-off is controlled by the parameter $0 \le \lambda$.

a) Assuming an investor population of mean-variance optimizers, derive an expression for the market (*i.e.*, tangent) portfolio.

b) Given that different investors have different return goals or risk preferences, explain how an investor uses cash and the market portfolio to achieve them.

c) Explain why the approach you described in (b) above is superior in mean-variance terms to any other strategy.

6. Marchenko-Pastur Distribution

You are given the returns of $N = 100$ assets over $T = 250$ time periods.

a) Compute the parameter q for the Marchenko-Pastur distribution of eigenvalues for a correlation matrix of uncorrelated assets for an estimation problem of this type.

b) Compute the lower and upper bound for the associated Marchenko-Pastur distribution given q.

c) You are given the partial list of sorted eigenvalues of the sample correlation matrix: {15.2, 8.2, 4.2, 3.1, 2.8, 2.2, 1.8, 1.6, 1.5, 1.4, 1.3, …}. Based solely on the distribution (without any adjustment for sample size), which eigenvalues appear to be statistically meaningful?

7. Kendall's Tau of Gaussian Copula

Let $X = (X_1, X_2)'$ be a bivariate Gaussian copula with correlation $\frac{\sqrt{3}}{2}$ and continuous margins.

Show that the Kendall's τ is:

$$\rho_\tau = \frac{2}{3}$$

8. VaR in ARCH Model

Consider the following AR(1)-ARCH(1) model for the daily return r_t of an asset:

$$r_t = \theta r_{t-1} + u_t \qquad u_t = \sigma_t \varepsilon_t \qquad \sigma_t^2 = \omega + \alpha u_{t-1}^2$$

where $-1 < \theta < 1$, $\omega > 0$ and $\alpha \in (0,1)$.

What is the 99% 2-day VaR of a long position at time t ?

9. Risk Premium

The *beta*, denoted by β_i , of a risky asset *i* that has return r_i is defined by:

$$\beta_i = \frac{\mathrm{Cov}(r_i, r_M)}{\sigma_M^2}$$

where r_M is the market return and σ_M is its standard deviation.

The capital asset pricing model (CAPM) relates the expected excess return (also called the risk premium) $\mu_i - r_f$ of the asset *i* to its *beta* via:

$$\mu_i - r_f = \beta_i\,(\mu_M - r_f) \tag{1}$$

where r_f is the risk-free rate and $\mu_M - r_f$ is the market expected excess return.

The security market line refers to the linear relationship (1) between the expected excess returns $\mu_i - r_f$ and $\mu_M - r_f$. Derive equation (1).

10. Black-Scholes-Kolmogorov Equation

Consider a stock following a log-normal process:

$$dS_t = (r - q)\, S_t\, dt + \sigma S_t\, dW$$

1. Compute the probability distribution of $x_T = \mathrm{Ln}\left(\frac{S_T}{S_t}\right)$ given S_t.
2. Prove the Black-Scholes formula for a Call option with maturity T and strike K :

$$C(t, S, K, T) = e^{-r(T-t)}\,(F\, N(d_1) - K\, N(d_2)) \qquad (1)$$

Where $F = e^{(r-q)(T-t)}S$ is the forward, $d_1 = \frac{Ln(F/K)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$

$N(x) = \int_{-\infty}^{x} g(s)ds$ is the normal cumulative function, with $g(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$

3. Compute the "Greeks":

$$\Delta = \frac{\partial C}{\partial S} \qquad \Gamma = \frac{\partial^2 C}{\partial S^2} \qquad \theta = \frac{\partial C}{\partial t}$$

4. Prove that C satisfies the Black-Scholes-Kolmogorov P.D.E.

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-q)S\frac{\partial C}{\partial S} - rC = 0 \qquad (2)$$

If you are short in time, you may *either* do 1 and 2 only (i.e. prove Black-Scholes formula (1)) *or* admit formula (1) and do 3 and 4 (i.e. prove Black-Scholes equation (2))

11. Ornstein-Uhlenbeck Process

We consider a stochastic process X_t with mean-reversion:

$$dX_t = \alpha\,(c - X_t)\,dt + \sigma\,dW$$

Such a process, called Ornstein-Uhlenbeck, is known for having a Gaussian distribution, which we would like to compute.

1. Given a function $f(x)$ satisfying $\frac{df}{dx} = \beta - \alpha f(x)$, compute $f(x)$ with respect to $f(0)$, α, β and t.
2. Find a differential equation satisfied by $m_t = \mathrm{E}(X_t)$ and compute m_t with respect to X_0 and t. What is $\lim_{t \to +\infty} m_t$?
3. Find a differential equation satisfied by $v_t = \mathrm{Var}(X_t)$ and compute v_t with respect to X_0 and t. What is $\lim_{t \to +\infty} v_t$?

12. Doob Theorem

Doob-Meyer decomposition theorem of a semi-martingale into a martingale and the rectifiable process has a discrete-time version, initially due to Joseph Doob (published in 1953, before Meyer's generalization to continuous time processes in 1962-63)
Consider a filtration $(\mathfrak{I}_t)_{t\in\mathbb{N}}$ with discrete time $t = 0,1,2\dots$ and a Markov process X_t .

We remind the *Markov property*: The conditional distribution of X_{t+1} knowing the previous value X_t is the same as that knowing the whole past $(X_s)_{s\leq t}$.

A process A_t is *predictable* if A_t is measurable with respect to $\mathfrak{I}_{t-1},$ in other words, its value is known using information until $t - 1$.

A process M_t is a *martingale* if it satisfies $E(M_{t+1} \mid \mathfrak{I}_t) = M_t$, in other words, if it has no drift.

Prove the following theorem:

Theorem (Doob): *Any Markov process (in discrete time) can be uniquely decomposed as:*

$$X_t = A_t + M_t$$

Where A_t is a predictable process and M_t is a martingale satisfying $M_0 = 0$.

Hint: Consider $E(X_t \mid \mathfrak{I}_{t-1})$

Scratch paper 1

Scratch paper 2

Scratch paper 3

Scratch paper 4

Scratch paper 5

Scratch paper 6

Scratch paper 7

Scratch paper 8

Scratch paper 9

Scratch paper 10

Scratch paper 11

Scratch paper 12

Scratch paper 13

Scratch paper 14

Scratch paper 15